Comments about *Bridges in Mathematics*

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In response to the request by the State Board of Education (SBE) to revisit the decision about *Bridges in Mathematics*, OSPI asked us to provide an analysis of the mathematical soundness of *Bridges*. In order to be consistent with the curriculum review report of Strategic Teaching, we examined three threads: multiplication of whole numbers, area of a triangle, and addition/subtraction of fractions. These are clearly critical aspects of any analysis of the mathematical soundness of a program, but they are not the only threads that might be chosen for examination, for example, addition/subtraction of whole numbers and measurement of length.

We have attempted to keep our analysis focused on mathematical soundness. We did not address the alignment of *Bridges* to the Mathematics Standards, nor did we examine the pedagogical approach of the authors of *Bridges*.

Multiplication of Whole Numbers

There are several Performance Expectations that are important for examining the mathematical soundness of this thread. It is important to address both conceptual and procedural aspects of this thread.

2.4.C Model and describe multiplication situations in which sets of equal size are joined.

3.2.A Represent multiplication as repeated addition, arrays, counting by multiples, and equal jumps on the number line, and connect each representation to the related equation.

3.2.D *Apply and explain strategies to compute multiplication facts to 10x10 and the related division facts.*

3.2.G Multiply any number from 11 through 19 by a single-digit number using the distributive property and place value concepts.

4.1.A *Quickly recall multiplication facts through 10x10 and the related division facts.*

4.1.C *Represent multiplication of a two-digit number by a two-digit number with place value models.*

4.1.D *Multiply by 10, 100, and 1,000.*

4.1.F *Fluently and accurately multiply up to a three-digit number by one- and two-digit numbers using the standard algorithm.*

4.1.I *Solve single- and multi-step word problems involving multi-digit multiplication and verify the solutions.*

<u>Grade 3</u>

The basis of mathematical soundness is laid in Grade 3, and we believe that it is laid quite well. Models for multiplication (e.g., arrays, place value models) lay the conceptual basis for the work on multiplication that the Mathematics Standards require for Grade 4. In particular, the chart of models of facts (e.g., p. 466 of the Teacher's Guide) seems critical for helping students see the basis of the commutative property of multiplication. This chart lays out key mathematical ideas in succinct ways that seem quite revealing of mathematical relationships. Too, the emphasis on common factors of pairs of numbers lays important mathematical groundwork for both division

and fractions. Indeed, investing time required for Unit 4 in Grade 3 seems time-well-spent for developing the mathematical soundness of multiplication.

Grade 4

In Unit 1, the discussion for teachers (pp. 5-6) shows clear models for the commutative and distributive properties. These models are also shared with students (see pp. 59, 66, and 74 of the Teacher's Guide and Session 13). The models also appeared in Grade 3 in the chart on p. 466 of the Teacher's Guide.

The use of an area model for multiplication of multi-digit numbers is helpful for presenting the mathematics of multiplication appropriately; for example, see Session 8 of Unit 1, and many of the sessions in Unit 2. The modeling of two-digit by two-digit numbers in Unit 2 builds on the distributive property of multiplication over addition. This work builds to the discussion of the standard algorithm on pp. 219-220 (Session 14). The four pieces of the array model are tied individually to the "partial products" recording scheme for the standard algorithm, but these array pieces are also regrouped to show the representation of the two partial products in the standard algorithm. Students may struggle with this regrouping, but that is a learning issue, not a mathematical soundness issue. We would argue that other number sentence representations that build off of the distributive property (e.g., 4 x 28 = 80 + 32 = 112) might be helpful for communicating the underlying mathematics, as well as for laying the groundwork for future algebraic reasoning, but this is for us a fairly minor issue. There may need to be more attention on how to record the "carrying part" of the algorithm, but this, again, is not a mathematical soundness issue.

In the "mainline" materials, there is less attention to standard algorithms than is expected in the WA Mathematics Standards. That presumably is why the authors provided supplementary lessons for WA. These supplementary lessons are part of the submission, so it seems safe to say that by the end of Grade 4, the standard multiplication algorithm has been taught appropriately. In supplement A5, recording of various versions of the algorithm is shown explicitly to students on student pages 34, 43, 79, 80 and 82, as well as on Independent Worksheets 113, 115, 117, 121, and 125. (The variance in page numbers is due to the interspersing of "teacher pages" in the supplement.) Whether some students may need more practice than is provided is an instructional issue that teachers will, of course, have to deal with. In short, however, we believe that the mathematical development in Grade 4 is strong.

Grade 5

One of the limitations of the array model for two-digit by three-digit multiplication is that there is no agreed-upon, two-dimensional model for 1,000 (i.e., 10 x 100). The standard model for 1,000 with base ten blocks is a three-dimensional model. At some point in mathematics instruction, students need to generalize what they have learned. Symbols do have meaning in mathematics, and students need to understand that meaning. Ultimately, symbols can support the development of conceptual understanding of advanced mathematics ideas. The development of multiplication in *Bridges* seems to lay excellent groundwork for students to move from two-digit by two-digit multiplication to two-digit by three-digit multiplication, even though there are not many opportunities for students to practice two-digit by three-digit multiplication in the

materials. Teachers might have to help students make the generalization step, but the mathematical development provided is certainly strong enough to support this.

One of the obvious limitations of the standard mathematical algorithm is that in some instances it is not as efficient as other strategies. For example, 57 x 99 is much more efficiently computed as 57x100 - 57, and we suspect that many mathematicians would choose NOT to use the standard algorithm for this computation. We hope that students will learn flexibility of thinking so that the first step in performing a computation is to decide whether the standard algorithm or some other strategy is more efficient.

Conclusions for Multiplication of Whole Numbers

It is important for students to understand the connections between the symbols for multiplication (i.e., the algorithms) and the representations of multiplication (e.g., place value models). This is particularly true for representations and symbols for commutativity and distributivity. The conceptual supports provided for students in *Bridges* is exactly what students need to see in order to imbue meaning in the symbols. We conclude that for this thread, *Bridges* is indeed mathematically sound.

Area of a Triangle

There are several Performance Expectations that are important for examining the mathematical soundness of this thread. It is important to address both conceptual and procedural aspects of this thread.

4.3.B Determine the approximate area of a figure using square units.

4.3.C *Determine the perimeter and area of a rectangle using formulas, and explain why the formulas work.*

4.3.F Solve single- and multi-step word problems involving perimeters and areas of rectangles and verify the solutions.

5.3.D *Determine the formula for the area of a parallelogram by relating it to the area of a rectangle.*

5.3.E *Determine the formula for the area of a triangle by relating it to the area of a parallelogram.*

5.3.F Determine the perimeters and areas of triangles and parallelograms.

5.3.I *Solve single- and multi-step word problems about the perimeters and areas of quadrilaterals and triangles and verify the solutions.*

Grade 2

Study of area begins in Grade 2. The Bridges approach to area by dissecting figures into congruent parts and adding them is mathematically sound. In fact this is the approach to area in Euclid's Elements and in geometry books for more than 2000 years since, and it seems an eminently appropriate approach for students in Grade 2. Also, the authoritative twentieth century foundation of geometry by Hilbert defines the concept of area by its fundamental properties, especially decomposition of figures into triangles. Mathematically, this approach introduces two of the key properties of area: that geometrical shapes have area and that areas can be added when a figure is dissected. It also has the advantage that it does not confuse the *concept* of area with the numerical *measurement* of area using area formulas. Area is an

important mathematics concept in and of itself; students must understand the concept of area as well develop skill at measuring area. The importance of the distinction between concept and skill shows up in the notorious confusion of American students about the difference between area and perimeter; if one only thinks of these quantities as some number associated to a shape, it is easy to see how confusion can arise. At first glance, one might think that introduction of area by counting squares would be an appropriate approach. But if one starts with this idea, it is not clear at all how to measure the area of anything but rectangles until much later in the story. And even the area of rectangles only comes into its own when students begin to explore multiplication. Of course the approach taken in *Bridges* includes square as one example of unit area, so area measurement by squares is included as part of the development while also providing important conceptual foundation.

When the area formula for a triangle is finally developed, it is not found by covering the triangle with squares but by dissecting it. Basing the definition of area on squares has the drawback of conflating the concept of area with the measurement of area, and thereby creating the potential for confusion. We argue that having multiple ways of comparing areas (and even computing area using various units at Grade 2) is valuable for developing mathematical soundness, and it is consistent with a mathematically rigorous development of area. The mathematics here is very sound.

<u>Grade 3</u>
Unit 3 is a geometry unit. There is a work here on dissection into congruent figures that also lays some groundwork for the development of the area formula for triangles and parallelograms, even though those topics are not directly addressed here.

Unit 4 contains a substantial development of the array model for multiplication of natural numbers. This unit is about multiplication and learning the multiplication table, not about area. However, there is clearly a close connection between this work and the computation of areas of rectangles in terms of unit squares. (This connection is, in fact, one of the reasons that area of rectangles and multiplication of whole numbers are focus areas in the same grade – Grade 4 – in the Mathematics Standards.) So Unit 4 helps to create connections between number and area that will be important later. The systematic development of area of rectangles is consciously placed later.

<u>Grade 4</u>
When authors try to make a mathematical definition of area there is a genuine mathematical problem. While the familiar basics of area are accessible to students, a truly rigorous definition of area is too complicated and advanced for K-12 students even at the high school level; defining area and proving its basic properties either involves a complex proof about dissecting polygons or an advanced proof using limits. So authors of textbooks at all but the most advanced college level assume that a concept of area exists and state the properties it should have (e.g., congruent figures have the same area, and the area of a figure that is dissected is the sum of the areas of the dissected figures). This seems to be the approach in *Bridges*. This does not mean that an informal definition cannot be helpful as an explanation of area, but we have to recognize that "simple" definitions (e.g., Area is the number of square units needed to cover a space.) are mathematically inadequate. For example, what does "cover" mean? And how does this help to

know that the area of a triangle is invariant when any of the three sides could be used as "the base"? The mathematics used here (e.g., using different kinds of units) is mathematically sound and introduces students informally to some of the ambiguities of the topic.

In Unit One, Session 20 is devoted to measuring perimeters (in inches) and areas (in square inches) of various rectangles, and in particular investigating how two rectangles can have the same perimeter and different areas. (This is one of the explicitly stated Performance Expectations.) This reinforces the use of unit squares in measuring area.

Unit Four develops further student understanding of area, if not specifically area of triangles. The Area Bingo in Session 10 leads student to compute areas on a geoboard, using the unit square on the geoboard as the unit. In addition Sessions 16 and 18 introduce surface area of simple solids.

The supplements also contain more lessons on measurement of area. D5 is devoted to measuring area in square centimeters (and comparing areas). D6 measures areas both in square inches and square feet and compares the two. In addition, it broadens the collection of shapes whose areas are measured from rectangles to "hexarights" (i.e., non-convex hexagons whose angles are all right angles). This uses the area formula for rectangles but gives a richer collection of areas than one gets from working with rectangles alone.

Grade 5

There are two mathematically equivalent ways to relate the formulas for the area of a rectangle and the area of a parallelogram. One is to dissect a parallelogram and reassemble it as a rectangle and one is to start with a rectangle and dissect it into a parallelogram. These are mathematically equivalent, though we acknowledge that other mathematicians might prefer one of these approaches over the other. The choice of which one to use is a pedagogical choice, not a mathematical soundness issue. The details of the lesson suggest a reason for the choice in *Bridges*. Starting with the rectangle allows the teacher have each student start with a 3 by 5 index card instead of having to cut out individual parallelograms to start with. This has some pedagogical advantages, but it also has mathematical advantages, since students all start with equally correct models of a rectangle. One can argue that going from the familiar rectangle to the less familiar parallelogram makes pedagogical sense (but that is an issue of pedagogy, not mathematical soundness). We note that later exercises do being with parallelograms and work back to rectangles. Once the area formulas for parallelograms and rectangles are developed, students can develop the formula for area of a triangle.

Conclusions for Area of a Triangle

In these materials it is extremely clear that the area of a rectangle is computed from squares by the usual product formula. The other areas are studied by taking area as an undefined term but using the additive property for dissections that is the key property of area as presented in books by highly regarded mathematicians since the time of Euclid. On the other hand, students are provided with ample work and practice on finding the area of rectangles and then on understanding how rectangle area leads to the area of parallelogram and triangles. In fact the stage is set very well by prior work dissecting shapes. In the supplements the formulas are developed on square grid paper; this approach seems quite general. We feel that these materials

are spot on in satisfying the Standards 4.3.C, 5.3.D and 5.3.E. As for 5.3.I, we would submit that the problems such as "Which is Bigger", the problems about "quilt squares," and the problems about changes in unit squares are examples of word problems that fulfill the requirements of this standard.

Addition and Subtraction of Fractions

There are several Performance Expectations that are important for examining the mathematical soundness of this thread. It is important to address both conceptual and procedural aspects of this thread.

3.3.A *Represent fractions that have denominators of 2, 3, 4, 5, 6, 8, 9, 10, and 12 as parts of a whole, parts of a set, and points on the number line.*

3.3.B Compare and order fractions that have denominators of 2, 3, 4, 5, 6, 8, 9, 10, and 12.

3.3.C Represent and identify equivalent fractions with denominators of 2, 3, 4, 5, 6, 8, 9, 10, and 12.

4.2.F Write a fraction equivalent to a given fraction.

4.2.G Simplify fractions using common factors.

5.2.A *Represent addition and subtraction of fractions and mixed numbers using visual and numerical models, and connect the representation to the related equation.*

5.2.E *Fluently and accurately add and subtract fractions, including mixed numbers.*

5.2.H *Solve single- and multi-step word problems involving addition and subtraction of whole numbers, fractions (including mixed numbers), and decimals, and verify the solutions.*

<u>Grade 3</u>

One major goal of the Grade 3 Mathematics Standards is to represent fractions. Consequently emphasizing fractions "of" something seems completely consistent with the requirement of the Standards. Mathematically it is critical that this kind of modeling be developed carefully as a basis for understanding fractions as numbers. It may be too much to expect such abstract work in Grade 3 (though this is, of course, a pedagogical issue, not a mathematical soundness issue).

<u>Grade 4</u>

Unit 3 begins with exploration of halves. Then "ruler fractions" are introduced, both as a representation and as a model for addition of fractions. This work sets the stage for use of the number line. There are several lessons on "egg carton" fractions (i.e., denominators of 2, 3, 4, 6, and 12) followed by fractions on the clock with the same denominators. Using colored tiles provide engagement of fractions of areas. The supplements extend the work somewhat on how to find equivalent fractions.

In Unit 6, there is some practice with equivalent fractions (Session 3), though the denominators are all powers of two. More attention to equivalent fractions appears in Session 12, and there is more extensive work in Grade 5. The connection of fractions to decimals (e.g., Sessions 9 and 13) seems appropriate and mathematically useful. In Session 18 students order fractions and decimals together; this would be an important idea that provides support for later work with ratios in middle grades.

Grade 5

In Grade 5, Unit 4, fractions are part of several lessons. Fractions parts are introduced (p. 568 of Teacher's Guide) first for fractions with denominators of 2, 4, 8, and 16, and then for denominator so 3, 6, and 12. Quilt Blocks (Session 14) and "egg carton" fractions (Sessions 17-19) reinforce relationships among equivalent fractions. Home Connection 36 (Session 12) explicit presents fractions on a number line, which might be viewed as an introduction to fractions as numbers. Fractions as numbers are further emphasized with measurement of length in U.S. customary units (i.e., denominators of 2, 4, and 8). There is extension to additional denominators (e.g., 7) in Session 23.

Grade 5, Unit 6, materials begin by developing the model of a clock for adding/subtracting fractions. The examples that can be supported by this model are admittedly somewhat limiting, but given the focus on denominators of 2, 3, 4, 5, 6, 8, 9, 10, and 12 in the Standards, this seems like a very appropriate mathematical model. It is mathematically sound, though there are other models that might have been included. The interpretation of a fraction as a quotient (Session 2) is especially helpful for future development of mathematical ideas; "fraction as quotient" is an important interpretation of fractions that is often not presented to students explicitly in elementary school. Session 3 contains a model that is somewhat problematic, even though the intent is clear. The "12-inch" ruler in the pictures (pp. 794-795 of Teacher's Guide) represents a length that is actually longer than 12 inches. It is somewhat problematic, therefore, to associate 1/6 with 2 inches in that picture. Fortunately, this picture does not appear to be included in the student materials; it is only in the Teacher's Guide. Session 4 includes work with ways to write equivalent fractions. This seems adequate and counters the complaint that the Grade 4 materials do not deal with equivalent fractions.

The supplementary lessons include attention to least common denominator (LCD) and work with a variety of denominators. Of course, approaching this topic through LCD is mathematically sound, even though it is "hard." The lack of a "formula" for adding/subtracting fractions does not seem to us to be a serious issue (and does not seem required by the Standards). Indeed, Grade 5 students might not have enough experience with variables to make good sense of the formula, but that, of course, is a pedagogical issue, not a mathematical soundness issue. The Mathematics Standards include additional attention to operations on rational numbers in Grade 6, and this might a more appropriate place to include formalizing addition/subtraction with a formula (an alignment issue). Adding/subtracting with least common multiple certainly lays the groundwork for the formula; extending the instruction to include the formula for Grade 5 students who are prepared to understand that formalism would not seem to be difficult.

Conclusions for Addition/Subtraction of Fractions

The materials might make more use of fractions on number lines, but there are other PEs in Grades 6-8 that also address this issue. Too, there may not be quite enough development of equivalent fractions with denominators other than 2, 4, 8, and 16, but there is some development (included Home Connection pages), and teachers could extend this work. (The repeated suggestion in the Standards to represent fractions with denominators of 2, 3, 4, 5, 6, 8, 9, 10, and 12 may artificially suggest to curriculum authors that these are the only ones that need to be included.) We believe that students would be prepared to learn the mathematics outlined in the standards for Grades 6-8.

Summary

The mathematics of *Bridges* is sound. This is not to say that the alignment of *Bridges* is perfect; as noted in the OSPI curriculum alignment study, there is room for improvement. Too, some people might not agree totally with the pedagogical approach of *Bridges*, but pedagogical issues need to be separated from an analysis of the mathematical soundness.

It is important to point out that *Bridges* lays the conceptual groundwork for mathematical development of ideas. This is important both for having mathematically sound materials and for helping students actually learn the intended mathematics. We acknowledge that mathematicians do not always have a single view about how mathematical ideas should be developed. Other mathematicians might disagree with the choices made in *Bridges*, but again, this is not really an issue of mathematical soundness. It is more an issue of philosophical approach to mathematics.